Fluid Statics

I. Pressure

A. Fluids can exert normal and shear forces on surfaces of contact, however if fluid is at rest relative to surface then viscosity will have no shearing effect

B. Pressure: Force acting normal to an area divided by this area

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

C. Units are Pascals ($Pa = N/m^2$), PSF (lb/ft^2), or PSI (lb/in^2)

D. Pascal's Law

1. Pascal's Law: Intensity of Pressure acting at a point in a fluid is the same in all directions
2. Since pressure at a point is transmitted throughout the fluid by action, equal but opposite force reaction, any pressure increase Δp at one point in fluid will cause same increase at all other points within fluid

II. Absolute and Gage Pressure

A. Zero Absolute Pressure: A container with nothing inside (Perfect Vacuum)

B. Absolute Pressure: Any pressure that is measured above zero absolute pressure

C. Standard Atmospheric Pressure: Absolute pressure measured at sea level and at a temperature 15°C. $P_{atm} = 101.3\ kPa$ (14.7 psi)

D. Gage Pressure: Any pressure measured above or below atmospheric pressure

$$P_{abs} = P_{atm} + P_g$$

III. Static Pressure Variation

A. Pressure varies in static fluids due to the weight of the fluid

B. Pressure does not change in the x and y directions; Pressure remains constant in the horizontal plane

C. Pressure will only be a function z

D. $dp = -\gamma dz$; Negative sign indicates pressure will decrease as one moves upward

E. Applies to both incompressible and compressible fluids

IV. Pressure Variation for incompressible fluids

A. γ is constant for incompressible fluids since volume doesn't change

$$\int_{p_0}^{p} dp = -\gamma \int_{z_0}^{z} dz$$

$$p = p_0 + \gamma(z_0 - z)$$

B. Reference level is usually established at the free surface of the liquid, $z_0 = 0$, and coordinate z is directed positive downward

$p = p_0 + \gamma h$ where h is distance from surface

C. If surface pressure equals atmospheric pressure, then γh represents gage pressure; $p = \gamma h$

D. Weight of water causes gage pressure to increase linearly as you go deeper

E. Pressure Head

1. Pressure Head (h): Indicates the height of a column of liquid that produces the gage pressure p

$$h = \frac{p}{\gamma}$$

V. Pressure Variation for Compressible Fluids

A. Specific Weight γ is not consistant throughout a gas

B. To integrate $dp = -\gamma dz$, we must express γ as a function of p. We do this using $p = \rho RT$ & $\gamma = \rho g$ giving $\gamma = \frac{pg}{RT}$

$$dp = -\gamma dz = \frac{-pg}{RT} dz \quad \text{or} \quad \frac{dp}{p} = \frac{-g}{RT} dz$$

C. Constant Temperature

1. If temperature throughout gas remains constant, we have:

$$\int_{p_0}^{p} \frac{dp}{p} = -\int_{z_0}^{z} \frac{g}{RT} dz$$

$$\ln \frac{p}{p_0} = -\frac{g}{RT}(z - z_0)$$

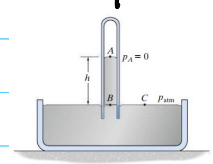
$$p = p_0 e^{-\left(\frac{g}{RT}\right)(z - z_0)}$$

2. This equation is used to calculate pressure within the lowest region of the stratosphere

VI. Measurement of Static Pressure

A. Barometer

1. Measures Atmospheric pressure
2. Closed end glass tube is filled with mercury and then submerged in a dish of mercury and turned upside down



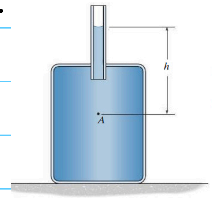
Simple barometer

3. Pressure at C equals pressure at B

$$p_B = p_A + \gamma_{Hg} h \quad ; \quad p_{atm} = 0 + \gamma_{Hg} h = \gamma_{Hg} h$$

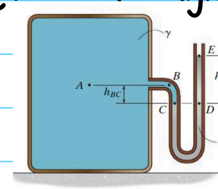
B. Manometer

1. Consists of transparent tube that is used to determine gage pressure in a liquid
2. Simplest type of manometer is called piezometer



Piezometer

3. Pressure at point A equals $p_A = \gamma h$
4. Piezometers do not work well for measuring large gage pressures, since h would be large and not good at measuring large negative pressures as air might leak into the container
5. U-tube manometers are useful when measuring negative gage pressure or moderately high pressures
6. One end of tube is connected to the vessel containing a fluid of specific weight γ and other end is open to atmosphere.
7. To measure relatively high pressures, a liquid with a high specific weight γ', such as mercury, is placed in the U-tube



Simple manometer

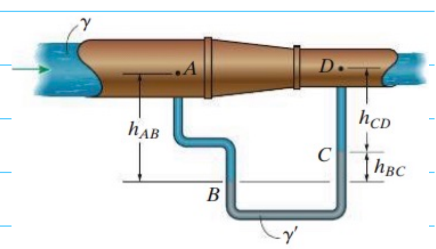
8. Pressure at A equals Pressure at B
9. Pressure at C equals $P_C = P_A + \gamma h_{BC}$
10. Pressure at C equals Pressure at D
11. $P_C = P_D = \gamma' h_{DE}$, thus $\gamma' h_{DE} = P_A + \gamma h_{BC}$ or $P_A = \gamma' h_{DE} - \gamma h_{BC}$

C. Manometer Rule

1. Manometer Rule: Start at a point in the fluid where the pressure is to be determined and proceed to add to it the pressures algebraically from one vertical fluid interface to the next, until you reach the liquid surface at the other end of the manometer
2. Pressure terms are positive if it is below a point

D. Differential Manometer

1. Used to determine the difference in pressure between two points in a closed fluid system
2. Summing pressures as outlined by manometer rule



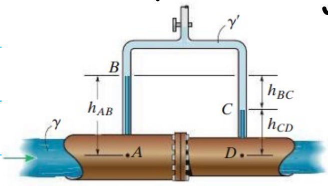
$$P_A + \gamma h_{AB} - \gamma' h_{BC} - \gamma h_{CD} = P_D$$
$$\Delta P = P_D - P_A = \gamma h_{AB} - \gamma' h_{BC} - \gamma h_{CD}$$

3. Since $h_{BC} = h_{AB} - h_{CD}$:

$$\Delta P = -(\gamma - \gamma') h_{BC}$$

4. Small differences in pressure can also be detected by using an inverted U-tube manometer. A manometer fluid with a smaller specific weight γ'

$$P_A - \gamma h_{AB} + \gamma' h_{BC} + \gamma h_{CD} = P_D$$
$$\Delta P = P_D - P_A = -\gamma h_{AB} + \gamma' h_{BC} + \gamma h_{CD}$$



5. Since $h_{BC} = h_{AB} - h_{CD}$:

$$\Delta P = -(\gamma - \gamma') h_{BC}$$

E. Bourdon Gage

1. Used for very high gage pressures
2. Consists of a coiled metal tube that is connected at one end to the vessel where the pressure is to be measured. The other end is closed so that when the pressure in the vessel is increased, the tube begins to uncoil and respond elastically

F. Pressure Transducers

1. Electromechanical device that can be used to measure pressure as a digital readout
2. Produces a quick response to changes in pressure, and provides a continuous readout over time

G. Other Pressure Gages

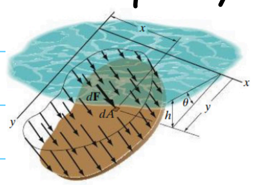
1. Fused quartz force-balance Bourdon tube: A more accurate gage; Pressure causes elastic deformation of a coiled tube that is detected optically. Tube is restored to original position by magnetic field that is measured and correlated to pressure
2. Piezoelectric gages: Change electric potential when subjected to small pressure changes, so pressure can be correlated to this change and presented as a digital readout

VII. Hydrostatic Force on a Plane Surface - Formula Method

A. Resultant Force

1. Resultant force is found by considering differential area dA at depth h; pressure at depth is $p = \gamma h$

$$dF = p\,dA = (\gamma h)\,dA = \gamma (y \sin\theta)\,dA$$
$$F_R = \gamma \sin\theta \int_A y\,dA = \gamma \sin\theta (\bar{y} A)$$



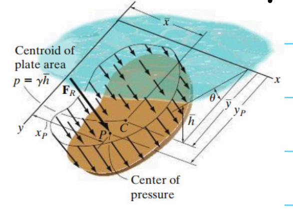
2. Integral $\int_A y\,dA$ represents moment of Area about the x axis
3. $\bar{y}$ represents distance from x-axis to the Centroid C
4. Depth of centroid $\bar{h} = \bar{y} \sin\theta$

$$F_R = \gamma \bar{h} A$$

5. Above equation shows that magnitude of resultant force on plate is product of pressure acting at plates centroid, $\gamma \bar{h}$, and area A of the plate

B. Location of Resultant Force

1. Resultant force of pressure distribution acts through a point called center of pressure
2. The y_p Coordinate



$F_R = \gamma \bar{h} A$
F_R acts through the center of pressure

   a) We require $(M_R)_x = \Sigma M_x$; $y_p F_R = \int_A y\, dF$

   b) Since $F_R = \gamma \sin\theta(\bar{y}A)$ and $dF = \gamma y \sin\theta\, dA$, then:

   $$y_p[\gamma \sin\theta(\bar{y}A)] = \int_A y[\gamma(y\sin\theta)\, dA]$$

   $$y_p \bar{y} A = \int_A y^2 dA$$

   c) The integral represents the area moment of inertia I_x for area about x-axis

   $$y_p = \frac{I_x}{\bar{y}A}$$

   d) Values for area moment of inertia are normally referenced from an axis passing through the Centroid of Area

   e) we use parallel-axis theorem to obtain I_x with this equation:

   $$y_p = \frac{\bar{I}_x}{\bar{y}A} + \bar{y}$$

3. The x_p Coordinate

   a) We use the same strategy to find the x_p Coordinate

   $$x_p[\gamma \sin\theta(\bar{y}A)] = \int x[\gamma(y\sin\theta)\, dA]$$

   $$x_p \bar{y} A = \int xy\, dA$$

   b) The above integral gives us the product of Inertia I_{xy} for the area

   $$x_p = \frac{I_{xy}}{\bar{y}A}$$

   c) We can apply parallel-plane theorem to get

   $$x_p = \frac{\bar{I}_{xy}}{\bar{y}A} + \bar{x}$$

   d) If either axis passes through the centroid, $\bar{I}_{xy} = 0$ and $\bar{x} = 0$ making $x_p = 0$ which means the center of pressure will lie on the $\bar{y}$ centroidal axis

VIII. Hydrostatic Force on a plane Surface - Geometrical Method

A. Resultant Force

1. The magnitude of the resultant force is equal to the total Volume of the "pressure prism"

   $$F_R = \Sigma F; \quad F_R = \int_A p\, dA = \int_{\forall} d\forall = \forall$$

B. Location

1. To locate resultant force, we require the moment of resultant force about the x and y axis to equal the moment created by the entire pressure distribution about these axes

   $$x_p F_R = \int x\, dF \qquad y_p F_R = \int y\, dF$$

2. Since $F_R = \forall$ and $dF = d\forall$, we have

   $$x_p = \frac{\int x p\, dA}{\int p\, dA} = \frac{\int x\, d\forall}{\forall} \qquad y_p = \frac{\int y p\, dA}{\int p\, dA} = \frac{\int y\, d\forall}{\forall}$$

3. The line of action of the resultant force will pass through both the centroid $C_{\forall}$ of the volume of the pressure prism and the center of pressure P on the plate

C) Plate having Constant Width

1. If a plate has constant width, then pressure loading along the width at depth h_1 and h_2 is constant
2. Intensity w of distributed load is measured as force/length and varies linearly from $w_1 = p_1 b = (\gamma h_1) b$ to $w_2 = p_2 b = (\gamma h_2) b$ where b = width
3. Magnitude F_R is then equivalent to the trapezoidal area defining the distributed loading

XI. Hydrostatic Force on a Plane Surface - Integration Method

A. Resultant Force

1. Considering a differential Area dA at a depth h where pressure is p, the resultant force is: $F_R = \int_A p\,dA$

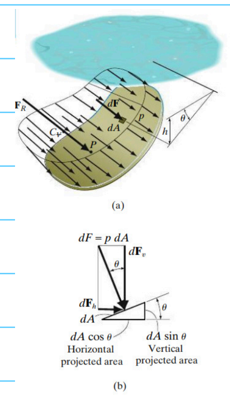
B. Location

1. Moment of F_R about x and y axes must equal moment of pressure distribution about these axes

$$x_P = \frac{\int_A \tilde{x}\, p\, dA}{\int_A p\,dA} \qquad y_P = \frac{\int_A \tilde{y}\, p\, dA}{\int_A p\,dA}$$

X. Hydrostatic Force on an inclined plane or curved surface determined by Projection

A. Horizontal Component



1. Force acting on dA is $dF = p\,dA$ so the horizontal component will be $dF_h = (p\,dA)\sin\theta$. We integrate to find resultant horizontal component
$$F_h = \int_A p \sin\theta\, dA$$
2. The resultant horizontal force component acting on the plate is equal to the resultant force of the pressure loading acting on the area of the vertical projection of the plate

B. Vertical Component

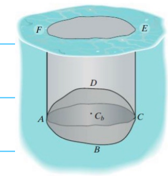
1. Vertical component of resultant force acting on dA is $dF_v = (p\,dA)\cos\theta$, so $dF_v = p(dA\cos\theta) = \gamma h (dA \cos\theta)$
2. Vertical Column above dA has a volume of $d\forall = h(dA\cos\theta)$, then $dF_v = \gamma d\forall$. Thus:
$$F_v = \int_{\forall} \gamma\, d\forall = \gamma \forall$$
3. Resultant Vertical Force acting on the plate is equivalent to the weight of the Volume of the liquid acting above the plate
4. Once vertical and horizontal components of force are known, the magnitude, direction, and line of action can be established for the resultant force, which acts through center of pressure

C. Gas

1. For gases, weight can generally be neglected, and so the pressure throughout is constant

XI. Buoyancy

A. Archimedes discovered the principal of buoyancy which is when a body is placed in a static fluid, it is buoyed up by a force that is equal to the weight of the fluid that is displaced by the body



Submerged body

B. The resultant force acting upward on the bottom surface (ABC) of a submerged body equals the weight of the fluid above the surface (ABCEFA)

C. The downward Force acting on the top surface (ADC) of the body equals the weight of the fluid above it (ADCEFA)

D. Difference in these two forces is known as the Buoyant Force which acts through the Center of Buoyancy, C_B, which is located at the Centroid of the Volume of liquid displaced by the body

E. Hydrometer

1. Uses buoyancy to measure specific gravity of a liquid
2. The hydrometer will float in equilibrium in water and will be marked specific gravity 1.0 at the water level, since $S_w = \gamma_w / \gamma_w = 1.0$
3. When the hydrometer is placed in another liquid it will float higher or lower depending on the liquids specific gravity compared to water.
   a) it will sink lower in lighter fluids

XII. Stability

A. A body can float in a liquid or a gas

B. Stable Equilibrium

1. If center of Gravity is below an objects center of Buoyancy, a moment will be created which keeps the object upright

C. Unstable Equilibrium

1. If center of gravity is above center of Buoyancy a moment is created which moves the object further from equilibrium

D. Neutral Equilibrium

1. Center of gravity and center of Buoyancy coincide, no matter the objects orientation, a moment will not be created

E. Metacenter: the point on the centerline of a body that intersects the line of action of F_b

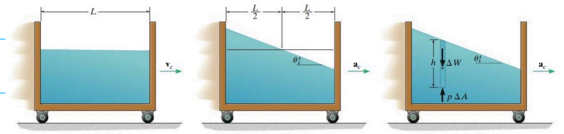
F. If the metacenter is above the center of gravity, the body will be in stable equilibrium

G. If metacenter is below center of gravity, the body will be in unstable equilibrium.

XIII. Constant Translational Acceleration of a Liquid

A. Constant Horizontal Acceleration

1. If a container of liquid has a constant velocity then the Surface of the liquid will remain horizontal
2. If a Container undergoes a constant acceleration, the liquid surface will begin to rotate clockwise about the center of the container and will eventually maintain a fixed tilted position θ



Constant velocity (a) Constant acceleration open container (b) Constant acceleration (c)

3. Vertical Element
   a) Equilibrium exists in vertical direction since no acceleration occurs in this direction
   $$\Sigma F_y = 0; \quad p\Delta A - \gamma(h\Delta A) = 0$$
   $$p = \gamma h$$
   b) Pressure at any depth from the inclined liquid surface is the same as if the liquid were static
4. Horizontal Element
   a) Only horizontal force is caused by the pressure of the adjacent liquid on each of its ends
   b) Since mass is $\Delta m = \frac{\Delta W}{g} = \frac{\gamma(x\Delta A)}{g}$
   $$\Sigma F_x = ma_x; \quad p_2\Delta A - p_1\Delta A = \frac{\gamma(x\Delta A)}{g}a_c$$
   $$p_2 - p_1 = \frac{\gamma x}{g}a_c \longrightarrow \frac{h_2 - h_1}{x} = \frac{a_c}{g}$$
   $$\tan\theta = \frac{a_c}{g}$$

B. Constant Vertical Acceleration

1. When accelerated upward, liquid surface maintains horizontal position, however pressure changes
2. Horizontal Element
   a) Pressure is the same for points that lie in the same horizontal plane
   $$\Sigma F_x = 0; \quad p_2\Delta A - p_1\Delta A = 0$$
   $$p_2 = p_1$$
3. Vertical Element
   a) Forces acting on vertical element consist of elements weight and pressure force on its bottom
   b) mass is $\Delta m = \Delta W/g = \frac{\gamma(h\Delta A)}{g}$
   $$\Sigma F_y = ma_y; \quad p\Delta A - \gamma(h\Delta A) = \frac{\gamma(h\Delta A)}{g}a_c$$
   $$p = \gamma h\left(1 + \frac{a_c}{g}\right)$$

c) If free fall occurs, then $a_c = -g$ and gage pressure throughout liquid will be zero

XIV. Steady Rotation of Liquid

A. When a liquid is placed within a cylindrical container rotating with constant angular velocity ω, the liquid will rotate as well, eventually moving with no relative motion to itself. Particles that are closer to the axis will move slower than those farther away, causing the liquid surface to form a forced vortex

B. Constant angular rotation ω produces a pressure gradient in the radial direction due to radial acceleration of liquid particles

$$a_r = \omega^2 r$$

C. To study pressure gradient we look at a ring element having a radius r, thickness Δr, and height Δh. Pressures on inner and outer sides of ring are p and $p + \left(\frac{\partial p}{\partial r}\right)\Delta r$ respectively

D. mass of ring is $\Delta m = \frac{\Delta W}{g} = \frac{\gamma \Delta V}{g} = \frac{\gamma(2\pi r)\Delta r\,\Delta h}{g}$

$$\Sigma F_r = m a_r; \quad -\left[p + \left(\frac{\partial p}{\partial r}\right)\Delta r\right](2\pi r \Delta h) + p(2\pi r \Delta h) = -\frac{\gamma(2\pi r)\Delta r \Delta h}{g}\,\omega^2 r$$

$$\frac{\partial p}{\partial r} = \left(\frac{\gamma \omega^2}{g}\right) r$$

Integrating we obtain $\quad p = \left(\frac{\gamma\omega^2}{2g}\right) r^2 + C$

Since $\quad p = \gamma h \qquad h = \left(\frac{\omega^2}{2g}\right) r^2$

E. This equation represents a parabola. Specifically, the liquid as a whole forms a surface that describes a paraboloid of revolution